A Search for the Integer Solutions of the Diophantine Equations $x^3 \mp y^3 = n$

M.A.Gopalan
Professor, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India.

S.Vidhyalakshmi
Professor, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India.

R.Presenna
M.Phil Scholar, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India.

Abstract - We search for non-zero integers such that each integer is expressed as the difference or sum of two cubical integers.

1. INTRODUCTION

Any odd prime number p can be written as the sum of two squares if and only if it is of the form $p = 4k + 1$, where $k \in N$. Generally, number n can be represented as a sum of two squares if and only if in the prime factorization of n, every prime of the form $4k + 3$ has even exponent [1]. In [2,3], the Diophantine equation of the form $x^3 - y^3 = n$ has been considered for its non-zero integer solutions where n is an arbitrary non-zero integer. In particular, a few numerical examples are presented in [3]. In [4], the authors have considered the representation of any integer as the sum of two cubes to a fixed modulus. In [5], an intrinsic characterization of positive integers which can be represented as the sum or difference of two cubes is given. In this context, one may also refer [6]. These results motivated us to obtain general representation for n which can be written as the difference or sum of two cubical integers. It seems that the explicit representations for "n" as the sum or difference of two cubes are not presented earlier.

2. METHOD OF ANALYSIS

Let n be any non-zero integer. Let p, q be two divisors of n such that

$$n = pq \quad (1)$$

Case 1: Representation of n where $n = x^3 - y^3$

Consider the equation

$$x^3 - y^3 = n \quad (2)$$

which is equivalent to the following system of equations

$$x - y = p \quad (3)$$

$$x^2 + xy + y^2 = q \quad (4)$$

Eliminating x between (3) and (4), the resulting equation is

$$3y^2 + 3yp + p^2 - q = 0 \quad (5)$$

Treating (5) as a quadratic in y and solving for y, we have

$$y_1 = \frac{1}{6}(-3p + \sqrt{12q - 3p^2})$$

$$y_2 = \frac{1}{6}(-3p - \sqrt{12q - 3p^2})$$

In view of (3), the corresponding x values are given by

$$x_1 = \frac{1}{6}(3p + \sqrt{12q - 3p^2})$$

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$$x_2 = \frac{1}{6}(3p - \sqrt{12q - 3p^2})$$

Note that, after performing numerical computations, $(12q - 3p^2)$ is a perfect square when

i. $p = 2\propto, \quad q = \propto^2 + 3k^2$

ii. $p = 2\propto - 1, \quad q = 3(k^2 - k) + \propto^2 - \propto + 1$

Thus, the required values of x, y and n satisfying (2) are exhibited in the following table.

n	x_1	y_1
$2\alpha(\alpha^2+3k^2)$	$k+\alpha$	$k-\alpha$
	$-k+\alpha$	$-k-\alpha$
$(2\alpha-1)(3k^2-3k+\alpha^2-\alpha+1)$	$k+\alpha-1$	$k-\alpha$
	$\alpha-k$	$1-k-\alpha$

Case 2: Representation of n when $n = x^3 + y^3$

Consider the equation

$$x^3 + y^3 = n \tag{7}$$

which is equivalent to the system of equations

$$x + y = p \tag{8}$$

$$x^2 - xy + y^2 = q \tag{9}$$

Eliminating x between (8) and (9), the resulting equation is

$$3y^2 - 3py + p^2 - q = 0 \tag{10}$$

Treating (10) as a quadratic in y and solving for y, we have

$$\left.\begin{aligned} y_1 &= \frac{1}{6}\left(3p + \sqrt{12q - 3p^2}\right) \\ y_2 &= \frac{1}{6}\left(3p - \sqrt{12q - 3p^2}\right) \end{aligned}\right\} \tag{11}$$

In view of (7), the corresponding x values are given by

$$\left.\begin{aligned} x_1 &= \frac{1}{6}\left(3p - \sqrt{12q - 3p^2}\right) \\ x_2 &= \frac{1}{6}\left(3p + \sqrt{12q - 3p^2}\right) \end{aligned}\right\} \tag{12}$$

Employing (6), (11) and (12), the corresponding representations of n along with the required values of x and y are presented below.

i. $n = 2\alpha(\alpha^2 + 3k^2), \quad x = \alpha - k, \quad y = \alpha + k$

ii. $n = (2\alpha - 1)(3k^2 - 3k + \alpha^2 - \alpha + 1), \quad x = \alpha - k, \quad y = \alpha + k - 1$

3. CONCLUSION

In this paper, we have presented the general representation of an integer that can be expressed as the difference or the sum of two cubical integers. It is quite interesting and worth mentioning here that, the represents for N to be written as the difference or the sum of two cubical integers are the same. To conclude, one may attempt to find general representation for integers which can be expressed as the difference or the sum of two higher power (>3) integers.

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